

X-590-73-190

PREPRINT

NASA

66298

# A SEMI-ANALYTICAL METHOD OF COMPUTATION OF OCEANIC TIDAL PERTURBATIONS IN THE MOTION OF ARTIFICIAL SATELLITES

(NASA-TM-X-66298) A SEMI-ANALYTICAL  
METHOD OF COMPUTATION OF OCEANIC TIDAL  
PERTURBATIONS IN THE MOTION OF ARTIFICIAL  
SATELLITES (NASA) 23 p HC \$3.25  
2)

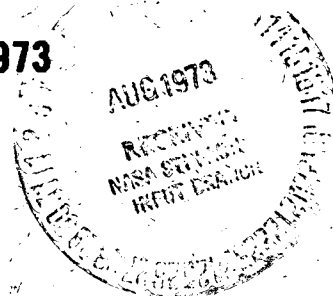
N73-27763

Unclas  
09912

CSCL 22C G3/31

PETER MUSEN

JULY 1973



**GODDARD SPACE FLIGHT CENTER**  
**GREENBELT, MARYLAND**

Reproduced by  
NATIONAL TECHNICAL  
INFORMATION SERVICE  
US Department of Commerce  
Springfield, VA. 22151

2400

**A SEMI-ANALYTICAL METHOD OF COMPUTATION  
OF OCEANIC TIDAL PERTURBATIONS IN THE MOTION  
OF ARTIFICIAL SATELLITES**

**Peter Musen**

**July 1973**

**Goddard Space Flight Center  
Greenbelt, Maryland**

## BASIC NOTATIONS

- $i, j, k$  - the basic unit vectors in the equatorial system, the x-axis is directed toward the vernal equinox and z-axis coincides with the axis of rotation of the Earth
- $G$  - the gravitational constant
- $M$  - the mass of the Earth
- $\theta$  - the sidereal time at Greenwich
- $\mathbf{r}'$  - the mean geocentric position vector of a water particle
- $\mathbf{r}$  - the geocentric position vector of the satellite
- $\Delta = |\mathbf{r} - \mathbf{r}'|$  - the distance between the satellite and the mean position of the particle of water
- $\mathbf{r}^0$  - the unit vector in the direction of  $\mathbf{r}$
- $\lambda, \mu, \nu$  - the equatorial components of  $\mathbf{r}^0$  in the equatorial system
- $\chi', \phi'$  - the east longitude and the latitude of the water particle in the state of equilibrium
- $\alpha$  - the right ascension of the satellite
- $\delta$  - the declination of the satellite
- $\kappa$  - the density of oceanic water
- $\kappa_0$  - the mean density of the Earth
- $R$  - the radius of the mean level surface of the sea water
- $R\zeta(\chi', \phi', t)$  - the height of water tide
- $k'_n$  - Love numbers associated with the effects of loading of the mass of water

$a$  - the semi-major axis of the orbit of the satellite

$n$  - the mean motion of the satellite,  $n^2 a^3 = GM$

$a, e, \Omega, \pi, i$  - the mean elliptic elements of the satellite

$p = R/a$  - the parallactic factor of the satellite

$l$  - the mean anomaly of the satellite

$f$  - the true anomaly of the satellite.

# A SEMI-ANALYTICAL METHOD OF COMPUTATION OF OCEANIC TIDAL PERTURBATIONS IN THE MOTION OF ARTIFICIAL SATELLITES

## INTRODUCTION

The problem of tidal influence on the motion of artificial satellites has captured the imagination of theoreticians and observers.

This interest was stimulated by the fact that from the observed magnitude of the tidal effects on the motion of satellites one can determine the elastic response of the earth as it is "seen" by the satellite from outer space.

The pioneering theoretical and computational work in this domain was done by Kozai (1965), Newton (1968) and Kaula (1969). The problem is comparatively simple, theoretically and numerically, if only the influence of the "solid Earth" tides on the motion of the satellite is to be taken into consideration. A number of authors (Smith et al., 1971), (Anderle, 1971), (Douglas et al., 1972) have treated this problem with success.

The electronic computer was applied recently to develop a semi-analytical theory of the Solid Earth tidal effects (Musen and Estes, 1972), (Musen and Felsentreger, 1973), in the form of Fourier series with purely numerical coefficients. However, recent observations indicate a disagreement with the theory and show an apparent dependence of elastic parameters of the Earth

(Love numbers) upon the orbital inclination of the satellite (Smith, 1973), (Douglas et al., 1972). The influence of the tides of the hydrosphere can be the primary cause of this disagreement and of the apparent dependence of Love numbers upon the inclination (Lambeck and Cazenave, 1973). It seems that the perturbations caused in the motion of satellites by the ocean tides can reach as much as 20% of the perturbations caused by the tides of the Solid Earth, and both kinds of perturbations have the same frequency spectrum. The most significant perturbations, which will affect the values of Love numbers, are caused by the ocean diurnal tides  $K_1$ ,  $O_1$ ,  $P_1$  and by semi-diurnal tides  $M_2$ ,  $S_2$  and  $P_2$ . The  $K_1$ -tide contributes to the largest periodic term in the expansion of perturbations in the orbital inclination and thus influences the value of Love number  $k_2$ . In addition, the influence of normal modes of particular oceans also deserves consideration. The central computational problem consists of the determination of amplitudes and lags of the satellite's perturbations from the amplitudes and lags of the tidal constituents as exist over the surface of the world ocean.

In the present article we discuss the method of expansion of the satellite's perturbations, as caused by the oceanic tides, into Fourier series with the arguments  $\Omega$  and  $\pi$  of the satellite and  $\ell$ ,  $\ell'$ ,  $F$ ,  $D$  and  $\Gamma$  of the Moon. The coefficients in this expansion are purely numerical and peculiar to each particular satellite. Such a method is termed as semi-analytical in Celestial Mechanics. Gaussian form of the differential equations for variation of elements, with the right hand sides averaged over the orbit of the satellite, is convenient to use with the semi-analytical expansion.

## EXPANSION OF THE DISTURBING FUNCTION

The disturbing function associated with the direct attraction of the satellite by the tidal shell of water has the form:

$$U = G\kappa R^3 \iint_S \frac{\zeta(\chi', \phi', t)}{\Delta} d\sigma' \quad (1)$$

where

$$d\sigma' = \cos \phi' d\phi' d\chi',$$

$$\zeta = O(10^{-7}).$$

The effects of loading on the bottom of the sea (Hendershott, 1972) are temporarily omitted and will be introduced at the next stage. The double integration in (1) is performed over the mean surface of the global ocean (or over the mean surface of a particular sea whose tidal influence on the satellite we want to determine).

Introducing the mean density of the Earth

$$\kappa_0 = \frac{3M}{4\pi R^3}$$

we re-write (1) in the form:

$$U = 3 \frac{GM}{R} \cdot \frac{\kappa}{\kappa_0} \cdot \frac{1}{4\pi} \iint_S \frac{R}{\Delta} \zeta(\chi', \phi', t) d\sigma' \quad (2)$$

Substituting the expansion

$$\frac{R}{\Delta} = \sum_{n=0}^{+\infty} P^{n+1} \left( \frac{a}{r} \right)^{n+1} \sum_{m=0}^n N_{nm} P_{nm}(\sin \delta) P_{nm}(\sin \phi') \cos m(\alpha - \theta - \chi')$$

$$N_{no} = +1, \quad N_{nm} = 2 \cdot \frac{(n-m)!}{(n+m)!} \quad \text{for } m \neq 0$$

into (2) and introducing Love numbers  $k'_n$  to account for the effects of loading (Lamb, 1945), we obtain for the oceanic tidal disturbing function:

$$\dot{V} = 3 \frac{\kappa}{\kappa_0} \cdot \frac{GM}{R} \sum_{n=0}^{+\infty} P^{n+1} (1 + k'_n) \left( \frac{a}{r} \right)^{n+1} \quad (3)$$

$$\cdot \sum_{m=0}^n N_{nm} P_{nm}(\sin \delta) \cdot \frac{1}{4\pi} \iint_S \zeta(\chi', \phi', t) P_{nm}(\sin \phi') \cos m(\alpha - \theta - \chi') d\sigma'$$

We assume that the tidal oscillations of the sea at a given point  $(\chi', \phi')$  of the mean sea surface can be represented as a sum of periodic constituents of the form:

$$z(\chi', \phi', t) = Z(\chi', \phi') \cos[(\omega t + \nu) - s(\chi', \phi')].$$

$$\zeta = \sum z, \quad (4)$$

the phase angle  $s(\chi', \phi')$  is the retardation of the maximum ("high water") of  $z(\chi', \phi', t)$  at  $(\chi', \phi')$  relative to its high water at Greenwich. The basic data plotted on the geographic maps of a constituent are the isochrones of its retardation (the co-tidal lines) and the lines of equal amplitudes (the co-range lines).

From these data one can compute the lag and the amplitude of the corresponding tidal term in the perturbations of the satellite. To simplify the formulas we represent a constituent not in the standard form (4) but rather in an "old fashioned" manner as:

$$z(\chi', \phi', t) = Z(\chi', \phi') \cos[m(\theta + \chi') + q(\chi', \phi') + \psi]. \quad (5)$$



where

$$\psi = \sigma t + \beta$$

and  $\sigma$  is small. The local sidereal time  $\theta + \chi'$  disappears from the disturbing function in the process of elimination of the short period terms; the slowly changing part  $\psi + q$  remains.

The most significant long period effects are produced in the motion of satellites by diurnal ( $m = 1$ ) and semi-diurnal oceanic tides ( $m = 2$ ). Among the diurnal constituents  ${}^m K_1$  and  ${}^s K_1$  (combined together into  $K_1$ ),  $O_1$  and  $P_1$  are the most significant ones.  $K_1$  produces in the perturbations of the satellite terms with the argument  $\Omega$ . Such a term has the largest amplitude in the tidal perturbations of  $i$  as caused by the solid Earth and it is proportional to Love number  $k_2$ . The contribution by  $K_1$  is of special significance in the case of satellites with very high inclination (Isis type). It is clear, therefore, that the oceanic tides with the period of one sidereal day affect considerably the determination of  $k_2$ . Two other constituents,  $O_1$  and  $P_1$ , also have their share in modifying the value of  $k_2$ . However, their contributions will have small amplitudes and shorter periods than the one previously mentioned. The arguments  $\psi$  associated with  $O_1$  and  $P_1$  are  $2\ell' + 2D + 2\Gamma$  and  $2\ell' + 2\Gamma$ , respectively. The computation of the influence of the oceanic  $K_1$ -constituent is facilitated by the presence of large amphidromies in the North Atlantic and the Pacific. The  $K_1$ -tides are highest in the Pacific and the Indian Ocean (Defant, 1961).

In order to fully understand the structure of the  $\Omega$ -term in the tidal perturbations of the satellite the numerical integration of the Laplace tidal differential equations shall be repeated for  $K_1$  over the World Ocean to check or to improve Dietrich's results (1944). This integration shall be performed in a manner similar to one used recently for  $M_2$  and  $S_2$ , taking coastal boundary conditions and friction into consideration. Besides the real tides, the oceans are also subjected to free oscillations. For example, the North Atlantic has free oscillations with the periods of 21<sup>h</sup>.2, 14<sup>h</sup>.0 and 11<sup>h</sup>.5, with one, two and three amphidromic systems, respectively (Platzman, 1972). Oscillations of the sea such as these, with the periods of approximately a day or half-a-day, can also produce long period perturbations in the motion of the satellite.

We substitute (5) into (3) and omit from the result all of the terms which contain local sidereal time in the argument. The resulting disturbing function is free from the short period perturbations caused by the Earth rotation (Lambeck and Cazenave, 1973). We have for a given constituent:

$$V_m = \frac{3}{2} \cdot \frac{GM}{R} \cdot \frac{\kappa}{\kappa_0} \sum_{n=m}^{+\infty} V_{nm} \quad (6)$$

where

$$V_{nm} = p^{n+1} (1 + k'_n) A_n N_{nm} \left( \frac{a}{r} \right)^{n+1} P_{nm}(\sin \delta) \cos(m\alpha + \psi_n), \quad (7)$$

$$\psi_n = \sigma t + (\beta + \epsilon_n),$$

and

$$A_n \cos \epsilon_n = \frac{1}{4\pi} \iint_S Z(\chi', \phi') P_{nm}(\sin \phi') \cos q(\chi', \phi') d\sigma' \quad (8)$$

$$A_n \sin \epsilon_n = \frac{1}{4\pi} \iint_S Z(\chi', \phi') P_{nm}(\sin \phi') \sin q(\chi', \phi') d\sigma' \quad (9)$$

$$n = m, m + 1, \dots$$

As pointed out already we suggest only the computation of the influence of diurnal ( $m = 1$ ) and of semi-diurnal tides ( $m = 2$ ). The amplitudes  $A_n$  and the lags  $\epsilon_n$  are different for each constituent. In order to simplify the notations and to avoid the use of multiple indices we use in (8)–(9) the same notations for a given  $n$ , independently of  $m$  and of the constituent. The formulas (8)–(9) permit one to evaluate the amplitudes and lags of an oceanic tidal component in the perturbations of the satellite from the amplitudes and lags distributed over the surface of the ocean. The existence of amphidromic systems facilitates greatly this evaluation. The necessary basic information can be taken from the charts of the co-tidal and co-range lines. Such charts, obtained recently by integrating numerically Laplace tidal differential equations over the global ocean, are given for  $M_2$  and  $S_2$  components (without the effect of friction) by Bogdanov and Magarik (1967), for  $M_2$ , including friction (mainly along the costs) by Pekeris and Accad (1969) and, most recently by Hendershott (1972). For  $K_1$  the chart was given by Dietrich (1944).

We can express the spherical harmonics

$$Y_{nm} = P_{nm}(\sin \delta) \cos(m \alpha + \psi_n)$$

in (7) in terms of the components  $\lambda, \mu, \nu$  of the geocentric unit vector of the satellite by using the general formula

$$e^{im\alpha} P_{nm}(\sin \delta) = \frac{(n+m)!}{2^m m! (n-m)!} (\lambda + i\mu)^m \sum_{j=0}^{[(n-m)/2]} a_j (\lambda^2 + \mu^2)^{2j} \nu^{n-m-2j}, \quad (10)$$

where

$$a_{j+1} = - \frac{(n-m-2j)(n-m-2j-1)}{4 \cdot (j+1)(m+j+1)} a_j \quad (11)$$

$$= (-1)^{j+1} \frac{(n-m)(n-m-1) \dots (n-m-2j-1)}{2^{2j+2} \cdot (j+1)! (m+1)(m+2) \dots (m+j+1)}$$

$$j = 0, 1, \dots, \left[ \frac{n-m}{2} \right],$$

or by means of the recursive relation

$$(2n+1) \nu Y_{nm} = (n-m+1) Y_{n+1,m} + (n+m)(\lambda^2 + \mu^2 + \nu^2) Y_{n-1,m}. \quad (12)$$

Such expansions in terms of  $\lambda, \mu$  and  $\nu$  are convenient for the computation of the tidal perturbations in rectangular coordinates by means of numerical step by step integration.

However, the physical characteristics of the tidal perturbations, the relative importance of amplitudes and periods, as well as the presence of resonances, can be read and traced easier from a periodic expansion than from a numerical integration. For this reason we develop in the present work the theory of the oceanic tidal perturbations based on the expansion into trigonometric series with the arguments  $\pi$  and  $\Omega$  of the satellite and  $\ell$ ,  $\ell'$ ,  $F$ ,  $D$  and  $\Gamma$  of the Moon. The coefficients of the expansion are purely numerical. We make use of the Gaussian form of the differential equations for variation of elements so that the numerical values of the orbital inclination and eccentricity can be substituted from the outset. Such an approach to tidal perturbations, termed as "semi-analytical" in Celestial Mechanics, leads to a compact form of trigonometrical series with arguments linear in time. Operations with them can be handled efficiently on an electronic machine using the program developed at Goddard Space Flight Center by R. Estes.

Substituting the value of  $i$  into

$$\begin{aligned}\lambda &= + \cos^2 \frac{i}{2} \cos(f + \pi) + \sin^2 \frac{i}{2} \cos(f + \pi - 2\Omega), \\ \mu &= + \cos^2 \frac{i}{2} \sin(f + \pi) - \sin^2 \frac{i}{2} \sin(f + \pi - 2\Omega), \\ \nu &= \sin i \sin(f + \pi - \Omega).\end{aligned}\tag{13}$$

We compute the semi-analytical expansions of

$$Y_{11} = \lambda \cos \psi - \mu \sin \psi$$

$$Y_{21} = 3\lambda\nu \cos \psi - 3\mu\nu \sin \psi$$

and of

$$\begin{aligned} Y_{22} &= 3(\lambda^2 - \mu^2) \cos \psi - 6\lambda\mu \sin \psi \\ Y_{32} &= 15\nu(\lambda^2 - \mu^2) \cos \psi - 30\lambda\mu\nu \sin \psi \end{aligned} \quad (15)$$

into finite Fourier series with the arguments  $f, \pi, \Omega$  and  $\psi$ . Making use of (12)

we can then compute step by step the semi-analytical Fourier expansions of

$$Y_{31}, Y_{41}, \dots$$

and

$$Y_{42}, Y_{52}, \dots$$

with the same arguments as before and numerical coefficients peculiar to each particular satellite.

We can set

$$\lambda^2 + \mu^2 + \nu^2 = 1$$

in (12) when we compute  $Y_{nm}$ , but we must keep this factor in the process of developing the recursive formula for the derivatives and for the force component normal to the orbital plane. To compute the general tidal perturbations normal to the orbital plane of the satellite we need the Fourier expansion of

$$Z_{nm} = \mathbf{R} \cdot \nabla_0 Y_{nm}$$

where  $\nabla_0$  is the gradient operator with respect to  $\mathbf{r}^0$ ,

$$\nabla_0 = \mathbf{i} \frac{\partial}{\partial \lambda} + \mathbf{j} \frac{\partial}{\partial \mu} + \mathbf{k} \frac{\partial}{\partial \nu}.$$

and

$$\mathbf{R} \cdot \nabla_0 = + \sin i \sin \Omega \frac{\partial}{\partial \lambda} - \sin i \cos \Omega \frac{\partial}{\partial \mu} + \cos i \frac{\partial}{\partial \nu}. \quad (16)$$

By applying the  $\nabla_0$  -operator to (12) and setting  $\lambda^2 + \mu^2 + \nu^2 = 1$  after the differentiation, we obtain:

$$\begin{aligned} & (2n + 1) \nu \nabla_0 Y_{nm} + (2n + 1) \mathbf{k} Y_{nm} \\ & = (n - m) \nabla_0 Y_{n+1,m} + 2(n + m) \mathbf{r}^0 Y_{n-1,m} \\ & + (n + m) \nabla_0 Y_{n-1,m}. \end{aligned}$$

Multiplying both sides of the last equation by  $\mathbf{R}$  and taking

$$\mathbf{k} \cdot \mathbf{R} = \cos i, \quad \mathbf{r}^0 \cdot \mathbf{R} = 0$$

into account we deduce the recursive relation

$$\begin{aligned} & (2n + 1) \nu Z_{nm} + (2n + 1) Y_{nm} \cos i \\ & = (n - m) Z_{n+1,m} + (n + m) Z_{n-1,m} \end{aligned} \quad (17)$$

Making use of (14)-(16) we obtain

$$\begin{aligned} Z_{11} &= \sin i \sin(\delta + \psi) \\ Z_{21} &= 3\nu Z_{11} + 3Y_{11} \cos i \end{aligned} \quad (18)$$

and

$$\begin{aligned} Z_{22} &= 6\lambda \sin i \sin(\delta + \psi) + 6\mu \sin i \cos(\delta + \psi) \\ Z_{32} &= 5\nu Z_{22} + 5Y_{22} \cos i \end{aligned} \quad (19)$$

Substituting the value of  $i$  for a given satellite and making use of (13) we expand (18)-(19) on the computer into Fourier series and making use of (17) obtain step by step the semi-analytical expansions of

$$Z_{31}, Z_{41}, \dots$$

and

$$Z_{42}, Z_{52}, \dots$$

After all necessary expansions are completed  $\psi$  must be replaced by the proper value of  $\psi_n$ . The formal accuracy of  $\lambda$ ,  $\mu$  and  $\nu$ , with which we start the computations, is  $10^{-7} - 10^{-8}$ . Because of the smallness of  $A_n$  the final accuracy we need in the expansion of  $Y_{nm}$  and  $Z_{nm}$ , is only about  $10^{-3} - 10^{-4}$ . This indicates that the recursive relations for  $Y_{nm}$  and  $Z_{nm}$  serve our needs quite well, without a drastic loss of accuracy for a considerable number of steps.

Unfortunately, we cannot say yet with complete security how many terms we require in the expansion of the disturbing function (7). The computations for particular satellites and their comparison with observations will show this more clearly. In connection with this point we wish to emphasize again the advantage of a semi-analytical way of treating the oceanic tidal perturbations of the satellite. The addition of any new term to the disturbing function can be done automatically and the decision about the necessary accuracy is left totally to the machine.

## DIFFERENTIAL EQUATIONS FOR VARIATION OF ELEMENTS

At the present time we are able to observe only the long period oceanic tidal perturbations in the elements of the satellite. In the present exposition we make use of the Gaussian form of the differential equations for perturbations in elements and eliminate the short period perturbations by averaging the right hand sides of the equation over the orbit of the satellite:

$$\frac{d\delta e}{dt} = -\frac{\sqrt{1-e^2}}{na^2e} \cdot \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial \Omega}{\partial f} dl, \quad (20)$$



$$\begin{aligned} \frac{d\delta\pi}{dt} = & + \frac{\sqrt{1-e^2}}{n a^2 e} \cdot \frac{1}{2\pi} \int_0^{2\pi} \left\{ -a \frac{\partial \Omega}{\partial r} \cos f + \left( \frac{a}{r} + \frac{1}{1-e^2} \right) \frac{\partial \Omega}{\partial f} \sin f \right\} dl \\ & + 2 \sin^2 \frac{i}{2} \frac{d\Omega}{dt}, \end{aligned} \quad (21)$$

$$\sin i \frac{d\delta\Omega}{dt} = + \frac{1}{n a^2 \sqrt{1-e^2}} \cdot \frac{1}{2\pi} \int_0^{2\pi} r (\mathbf{R} \cdot \nabla \Omega) \sin(f + \pi - \Omega) dl, \quad (22)$$

$$\frac{d\delta i}{dt} = + \frac{1}{n a^2 \sqrt{1-e^2}} \cdot \frac{1}{2\pi} \int_0^{2\pi} r (\mathbf{R} \cdot \nabla \Omega) \cos(f + \pi - \Omega) dl \quad (23)$$

$$\frac{d\delta L}{dt} = - \frac{2}{n a^2} \cdot \frac{1}{2\pi} \int_0^{2\pi} r \frac{\partial \Omega}{\partial r} dl + (1 - \sqrt{1-e^2}) \frac{d\pi}{dt} + 2 \sqrt{1-e^2} \sin^2 \frac{i}{2} \frac{d\Omega}{dt}, \quad (24)$$

As a result of eliminating the short period terms the semi-major axis  $a$  is not affected by tides. It is sufficient to consider the typical form of the disturbing function:

$$\Omega = \frac{3}{2} \frac{GM}{R} \cdot \frac{\kappa}{\kappa_0} V_{nm}$$

or

$$\Omega = n^2 a^2 B_{nm} \left( \frac{a}{r} \right)^{n+1} Y_{nm} \quad (25)$$

where

$$B_{nm} = p^n (1 + k'_n) A_n N_{nm}$$

From (25) we have

$$\mathbf{r} \nabla \Omega = n^2 a^2 B_{nm} \left( \frac{a}{r} \right)^{n+1} [-(n+1) \mathbf{r}^0 Y_{nm} + \mathbf{r} \nabla \mathbf{r}^0 \cdot \nabla_0 Y_{nm}].$$

Taking

$$\nabla \mathbf{r}^0 = \frac{1}{r} (\mathbf{I} - \mathbf{r}^0 \mathbf{r}^0) \quad \text{and} \quad \mathbf{r}^0 \cdot \nabla_0 Y_{nm} = n Y_{nm}$$

into account we obtain:

$$\mathbf{r} \nabla \Omega = n^2 a^2 B_{nm} \left( \frac{a}{r} \right)^{n+1} [\nabla_0 Y_{nm} - (2n+1) \mathbf{r}^0 Y_{nm}]. \quad (26)$$

and, as a consequence,

$$\mathbf{r} \mathbf{R} \cdot \nabla \Omega = n^2 a^2 B_{nm} \left( \frac{a}{r} \right)^{n+1} Z_{nm}. \quad (26')$$

Substituting (25) and (26) into the differential equations (20)-(24) we reduce them

to a form convenient for programming:

$$\frac{d\delta e}{dt} = - \frac{n \sqrt{1-e^2}}{e} B_{nm} \cdot \frac{1}{2\pi} \int_0^{2\pi} \left( \frac{a}{r} \right)^{n+1} \frac{\partial Y_{nm}}{\partial f} dl, \quad (27)$$

$$\begin{aligned} \frac{d\delta \pi}{dt} = & + \frac{n}{e \sqrt{1-e^2}} B_{nm} \cdot \frac{1}{2\pi} \int_0^{2\pi} \left( \frac{a}{r} \right)^{n+1} W_{nm} dl \\ & + 2 \sin^2 \frac{i}{2} \frac{d\delta \varphi}{dt}, \end{aligned} \quad (28)$$

where

$$W_{nm} = (n+1) M Y_{nm} + N \frac{\partial Y_{nm}}{\partial f},$$

and

$$M = \frac{1}{2} e + \cos f + \frac{1}{2} e \cos 2f,$$

$$N = 2 \sin f + \frac{1}{2} e \sin 2f,$$

$$\frac{d\delta\Omega}{dt} = \frac{n}{\sqrt{1-e^2} \sin i} B_{nm} \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{a}{r}\right)^{n+1} Z_{nm} \sin(f + \pi - \Omega) dl \quad (29)$$

$$\frac{d\delta i}{dt} = + \frac{n}{\sqrt{1-e^2}} B_{nm} \cdot \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{a}{r}\right)^{n+1} Z_{nm} \cos(f + \pi - \Omega) dl \quad (30)$$

$$\frac{d\delta L}{dt} = 2nB_{nm}(n+1) \cdot \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{a}{r}\right)^{n+1} Y_{nm} dl \quad (31)$$

$$+ \frac{1}{1 + \sqrt{1-e^2}} e \frac{d\delta\pi}{dt} + 2\sqrt{1-e^2} \sin^2 \frac{i}{2} \frac{d\delta\Omega}{dt} \quad (32)$$

The factors multiplying  $(a/r)^{n+1}$  in the integrands are finite Fourier series with the arguments  $f + \pi$ ,  $\Omega$  and  $\psi$ . Thus, the typical integrals to be evaluated have the form:

$$\begin{aligned} & \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{a}{r}\right)^{n+1} \frac{\cos}{\sin} [k(f + \pi) + j\Omega + \psi] dl \\ & = X_0^{-n-1, k}(e) \cdot \frac{\cos}{\sin} (k\pi + j\Omega + \psi), \end{aligned}$$

where  $X_0^{-n-1,k}$  are Hansen coefficients (Tisserand, Vol. I). To compute these coefficients we can use the relation

$$X_0^{-n-1,k}(e) = \binom{n-1}{k} \cdot \left(\frac{e}{2}\right)^k (1 - e^2)^{-n+1/2} \quad (33)$$

$$\times F\left(\frac{k-n+1}{2}, \frac{k-n+2}{2}, k+1, e^2\right). \quad (33)$$

For satellites presently used for the evaluation of tidal effects, we have  $e < 0.07$

and, consequently, we need only few terms in the hypergeometric series (33).

If the eccentricity is very small then the presence of  $e$  as a "small divisor" in the right hand side of the differential equations might cause difficulty in expanding the semi-analytical theory of perturbations in elliptical elements. It is evident from (33) that such a difficulty can arise only if  $k = 0$ , i.e., if there are terms of the form

$$\left(\frac{a}{r}\right)^{n+1} \frac{\cos}{\sin}(j\Omega + \psi)$$

under the integral sign. The terms of this type disappear in (27) because of differentiation with respect to  $f$ . As a result a semi-analytical expansion of the tidal  $d\delta e/dt$  can always be arranged and in this case  $e$  is only a spurious "small divisor." It does not cause any real difficulty. The real difficulty associated with small  $e$  can arise only on developing the tidal  $d\delta\pi/dt$ . We can see from the form of  $M$ ,  $N$  and  $W_{nm}$  that  $e$  will appear as a real "small divisor" only if there is a term of the form

$$K_{\sin}^{\cos}(f + \pi + j\Omega + \psi)$$

in the expansion of  $Y_{nm}$ . This will happen only if  $n$  is an odd integer. For  $n$  even the semi-analytical expansion of the tidal  $d\delta\pi/dt$  always can be arranged. The arguments in the resulting series for derivatives of the elements are all linear in time and, consequently, the integration is straightforward (except in a case of a sharp resonance).

## CONCLUSION

In the present work we give a set of formulas necessary to expand the satellite's oceanic tidal perturbations in a semi-analytic form, as Fourier series with purely numerical coefficients, using a computer. All arguments are linear with respect to time (providing we do not have sharp resonances). This form permits one to compute the tidal perturbations and at the same time to make a judgement about the relative importance of amplitudes and periods of different terms. We select a semi-analytical, and not an analytical method, because of its compactness and because the basic information about the oceanic tides is given either in the semi-analytical form or in the form of charts of co-tidal and co-range lines.

On the basis of already existing information we can obtain a general estimate of the oceanic effects on the motion of an artificial satellite.

However, in order to fully understand these effects and to have a more accurate information about them we have to obtain more complete information about the global oceanic tides themselves.

We suggest a new numerical integration of Laplace tidal differential equations for those tidal constituents for which the work was done either on a limited basis or long ago.

In particular, it will be of great interest to obtain new information for  $K_1$ . This constituent contributes to the biggest amplitude in the expansion of the tidal perturbations of the orbital inclination. The corresponding term has the right ascension of the node as the argument.

The knowledge of oceanic perturbations in the motion of satellites will permit us to improve the values of the elastic parameters of the earth.

#### Acknowledgment

The author expresses his gratitude to his colleagues, Dr. D. Smith and Mr. W. Kahn, for stimulating and interesting discussions.

## REFERENCES

- Anderle, R. J., NWL Technical report No. TR-2889, U. S. Naval Weapons Laboratory, Dahlgren, Va. 1971
- Bogdanov, K. T., Magarik, V. A., Numerical Solution of the Problem of Propagation of Semi-diurnal Tidal Waves ( $M_2$  and  $S_2$ ) in the Global Ocean (Russ.) Doklady Akad. Nauk USSR Vol. 172, No. 6, pp. 1315-1317, 1967
- Defant, A., Physical Oceanography, Vol. II, Pergamon Press, 1961
- Dietrich, G., Die Schwingungssysteme der halb-und eintägigen Tiden in den Ozeanen., Veröff. Inst. Meeresk., Univ. Berlin, N. F. A. No. 41, 1944
- Douglas, B. C., Klosko, S. M., Marsh, J. G., Williamson, R. G., Tidal Perturbations of the Orbits of Geos-1 and Geos-2 NASA-X-553-72-475, 1972
- Hendershott, M. C., The Effects of Solid Earth Deformation on Global Ocean Tides, Geoph. J. R. As-Z Soc., 29, pp. 389-402, 1972
- Kaula, W. M., Tidal Friction with Latitude Dependent Amplitude, Astron. Journ. 74, pp. 1108-1114, 1969
- Kozai, Y., Effects of the Tidal Deformation of the Earth on the Motion of Close Earth Satellites, Publ. Astr. Soc. Japan 17, p. 395, 1965
- Lamb, H., Hydrodynamics, p. 305, Dover edition, New York, 1945
- Lambeck, K., Cazenave, A., Fluid Tidal Effects on Satellite Orbit and Other Temporal Variations in the Geopotential. G.R.G.S. No. 7, Jan. 1973

- Musen, P., Estes, R., On the Tidal Effects in the Motion of Artificial Satellites,  
Cel. Mech. 6, pp. 4-21, 1972
- Musen, P., Felsentreger, T., On the Determination of the Long Period Tidal  
Perturbations in the Elements of Artificial Earth Satellites, Cel. Mech. 7,  
No. 2, pp. 256-279, 1973
- Newton, R. R., A Satellite Determination of Tidal Parameters and Earth  
Deceleration, Geoph. Journ. Roy. Astr. Soc. 14, pp. 505-539, 1968
- Pekeris, C. L., Accad, Y., Solution of Laplace Equations for the  $M_2$  Tide in the  
North Oceans. Phil. Trans. A 265, pp. 413-436, 1969
- Platzman, G. W., North Atlantic Ocean, Preliminary Description of Normal  
Modes. Science 178, pp. 156-157, 1972
- Smith, D. E., Kolenkiewicz, R., Dunn, P. J., Geodetic Studies by Laser Ranging  
to Satellites, NASA-X-553-71-360, 1971
- Smith, D. E., Private Communication, 1973